Foreword

All school-age students need to develop a strong understanding of the essential concepts of mathematics to be able to expand professional opportunities, understand and critique the world, and to experience the joy, wonder, and beauty of mathematics. Mathematics learning occurs across grade levels, but an essential period of mathematics development is during middle school as students expand their learning beyond numbers to proportional reasoning which supports thinking algebraically. For some students, mathematics in middle school can be overwhelming and difficult, but school leaders and educators need to ensure that each and every student have access to meaningful mathematics curriculum and high-quality teaching for effective mathematics learning.

In middle school, mathematics teaching, and the process of learning algebraic readiness and proportionality, involve more than just acquiring content and carrying out procedures. At this level, students are expected to represent, analyze, and generalize about patterns. Students should be able to use multiplication and addition to find the relationship between the two sets of numbers and should look at patterns through the use of tables, graphs, and symbolic representation. Over time, with support from teachers, the mathematical practices and processes that students engage in as they engage with algebraic problems deepen their understanding of key concepts while developing procedural fluency.

Algebraic readiness and proportionality provide strong foundations for future mathematics courses. For students to be successful in algebra, it is essential that middle school mathematics teaching and learning provide opportunities to develop algebraic thinking and proportional reasoning. The strategies presented in *Teaching Math in Middle School: Using MTSS to Meet All Students' Needs* provide teachers with research-based ideas that will promote algebraic readiness for all students. Incorporating these concepts will provide students with the opportunity to experience success in middle school mathematics and in algebra.

Specifically, *Teaching Math in Middle School: Using MTSS to Meet All Students' Needs*, provides detailed information about using multi-tiered support systems (MTSS) to effectively teach mathematics to students who may experience difficulty with mathematics. This book is important for educators who need to teach a variety of learners in the classrooms and for school leaders and educators who want to put in place support systems that meet the needs of each and every learner.

Robert Q. Berry, III, Ph.D.
Professor, University of Virginia
President, National Council of Teachers of Mathematics.

Preface

As mathematics teachers, we wear our interest in and attraction to mathematics like badges, proud to tell everyone about how beautiful it is to learn about numbers, how they work, and how they help us understand the world around us. All four of us (the authors of this book) are a little geeky like that. In fact, combined we have more than 80 years of teaching and research interests in how students learn mathematics, how teachers teach mathematics, and how teachers and their colleagues can improve student learning in mathematics. We have all been science and mathematics teachers, teaching a range of topics including chemistry, biology, elementary mathematics, algebra, advanced algebra, trigonometry, physics, and calculus. In addition, we've all pursued graduate degrees focused on improving teaching and learning in mathematics (further evidence of geekiness). As you can see, we've invested a lot of our lives into improving the teaching of mathematics and the science areas that depend on knowledge of mathematics to make sense.

We decided to write this book for several reasons. First, we are struck by the evidence that being proficient in mathematics is key to academic success in life. Second, we believe that academic success should be accessible to everyone. Third, we have all observed our own students as well as others who believe that they are not capable of understanding and doing mathematics. Fourth, teachers who are prepared to teach mathematics well, working together with other education professionals, are the ingredient to ensuring student success in all subjects, but specifically in mathematics. We also believe that systems developed in schools, such as multi-tiered systems of support (MTSS/RTI), are important advances that will help students to succeed. And finally, we wrote this book because we are colleagues and friends who have learned a lot from one another over the years, and we hope that the information we have put in this book will help teachers and their education colleagues to improve their students' learning and confidence with middle school mathematics.

There is a growing body of research that suggests that students' mathematics achievement is an important predictor of later success in life as measured by educational and financial outcomes. Young children often start out in school with measurably different understandings of basic arithmetic, with children of color, children experiencing poverty, or children with disabilities at higher risk for poor mathematics achievement (Berch, Mazzocco, & Ginsburg, 2007; Hanusheck & Rivkin, 2006).

These performance differences are also remarkably stable with early school mathematics performance (K–1st grade) predicting later (5th grade) mathematics performance (Duncan et al., 2007). More recent findings suggest that knowledge of fractions and whole-number division, subjects taught in the intermediate grades, is more strongly related to high school math achievement than knowledge of whole-number addition, subtraction, and multiplication; verbal IQ; working memory; and parental income (Siegler et al., 2012). Taken together, these findings support several important notions about school mathematics: 1) helping students develop an understanding of mathematics early is critical to their later development, 2) development of understanding of rational numbers, in particular, has an important impact on students' later success, and 3) targeted interventions for students who struggle with particular areas of mathematics learning is necessary for their later success.

Because the evidence is abundantly clear that students' understanding of middle-level mathematics concepts (i.e., fractions and division) is critical to their development in higher level mathematics and their overall academic success, we feel it is particularly important that we ensure that all learners have access to high-quality mathematics teaching and the broadest range of instructional supports aimed at promoting their success. To achieve this, we first have to create a culture in schools and at home in which educators and parents believe that mathematics is *useful* and *learnable*. We need important figures in students' lives to promote their understanding of mathematics rather than promoting the idea that "some people are good at it" and "others are not math people." We also have to confront and change some educators' perceptions that some students can't learn mathematics and to recognize that there is evidence to support mathematical development for all students regardless of their background, early learning experiences, or challenges (e.g., Walker, 2007; Steele, 2003; Gersten et al., 2009).

Beyond the initial concern that mathematics is too difficult or too abstract for students to learn, teaching mathematics requires much more. Surveys show that many U.S. teachers at all grade levels have less extensive backgrounds in the mathematics they teach than is recommended by the National Council of Teachers of Mathematics. Still, most teachers feel comfortable with their mathematics content knowledge (Banilower et al., 2013). We would encourage all teachers to take a circumspect approach.

Strive to understand mathematics concepts and principles, be comfortable with your own knowledge, and feel confident that even those aspects of the discipline you find confusing can be learned if you persist in trying to understand. We certainly don't mean to portray this as an easy process. In fact, each of us has faced a time when we experienced our own "ah-hah" moments about a particular idea that we thought we had already mastered. For example, one of the core transitions in understanding that our students make in middle school mathematics is from whole numbers to rational numbers. Many of us experienced learning rational numbers with an approach that was mostly procedural and didn't maximize our understanding. As teachers, we continue to study how rational numbers function. It should not surprise us that we find ourselves asking fundamental questions about such things as dividing fractions, for example. Why is it that when you divide $\frac{1}{2}$ by $\frac{1}{2}$ you get a larger quotient? Rather than simply teaching an "invert and multiply" approach as we may have been taught, how do we encourage students to predict what will happen and then explain *why*?

It is also important that teachers feel competent and confident in doing mathematics and work hard to shrug off the idea that they have to know everything before teaching it. Many successful mathematics teachers deliberately build a culture in their classroom wherein making mistakes is considered necessary for learning. They model this behavior so that students feel comfortable taking risks in their problem solving and don't associate mathematics success with always being right.

In other words, teaching mathematics requires more than being able to do mathematics. Effective mathematics teachers understand how students conceptualize mathematics and how to develop their students' understanding in order to prepare for related concepts and principles that are on the horizon. They also develop their knowledge of common misconceptions students formulate that can disrupt their learning and how to diagnose those misconceptions. It takes experience and professional learning opportunities to develop these knowledge and skills that Ball, Hill, and Bass (2005) have referred to as "mathematics knowledge for teaching." We believe that this knowledge is particularly important when working with students who struggle to learn mathematics. The same survey we mentioned earlier about teachers' preparation in mathematics also reported that the vast majority of teachers do not feel that they have been adequately prepared to work with a diverse array of student needs in mathematics (Banilower et al., 2013).

From our perspective, making middle school mathematics accessible to all learners is a function of knowing your students' learning history, starting where they are, and designing instruction to help them grow in their knowledge and skills, tailoring instruction as needed to ensure that students develop proficiency in big ideas and providing appropriate accommodations when necessary for learners to continue to progress.

Our objectives in this book are to 1) set the context for the importance of supporting all learners in middle school mathematics, 2) share with you our understanding of effective instruction in order to build from a common vocabulary and understanding of the importance of teaching to learning, 3) examine the types of assessment necessary to ensure effective instruction and how different assessments assist teachers to support the full range of learners, and 4) offer ways of thinking about how teachers and other education professionals in a school or school district work collaboratively to optimize the positive impact of an MTSS/RTI approach to teaching mathematics.

HOW THIS BOOK IS ORGANIZED

Our book is structured in four sections. Section I, Building Numeracy in Middle School Students, introduces fundamentals to help math teachers instruct middle school students. Within Section I, Chapter 1, Laying the Foundation for Algebra, discusses the pillars of foundational knowledge middle-school students need to prepare for algebra. Chapter 2, Supporting All Students Through Multi-Tiered Instruction, introduces the widely used MTSS/RTI model. Chapter 3, Supporting All Students Through Differentiation, Accommodation, and Modification, introduces principles for tailoring instruction to meet all students' needs.

Effective implementation of MTSS depends on sound instructional methods and ongoing assessment. Section II of the book, Designing and Delivering Effective Mathematics Instruction, delves into best teaching practices. Within this section, Chapter 4, Aims for Effective Mathematics Instruction, presents overarching principles to guide teachers in planning and implementing lessons. Chapter 5, Evidence-Based Practices for Instruction and Intervention, grounds readers in research-supported teaching practices to use for core instruction in the general education classroom and for instructing students who need extra help. Chapter 6, Instructional Practices to Support Problem Solving, focuses on effective instruction related to problem solving, a common weakness for students with learning difficulties. Chapter 7, Designing Interventions, describes methods for creating and implementing effective intensive intervention. Finally, Chapter 8, Implementing Interventions Within a Multi-Tiered Framework, puts together information from the preceding chapters to explain how middle school math teachers can implement practical interventions and do so with fidelity.

In Section III, Using Data to Make Decisions, we guide teachers in using assessment results to inform instruction within MTSS/RTI. Chapter 9, Why Should We Assess?, provides an overview of the purposes of assessment and the different types of assessments used for each purpose—in essence, what questions we have about students' learning and how assessment helps us find answers. The remainder of Section III expands upon this overview, providing detailed guidance for conducting each type of assessment in Chapter 10, Who Needs Extra Assistance, and How Much? Universal Screeners; Chapter 11, Why Are Students Struggling? Diagnostic Assessments; Chapter 12, Is the Intervention Helping? Progress Monitoring; and Chapter 13, Have Students Reached Their Goals? Summative Assessments.

Successful implementation of MTSS/RTI depends not only on individual teachers' work in their own classrooms, but also on collaboration. Section IV of this book, Implementing MTSS to Support Effective Teaching, is written to help teachers collaborate effectively with other professionals and with parents. Chapter 14, MTSS in Action, guides educators through the details of planning instruction and assessment at each tier of intervention, and Chapter 15, Assessing Your School's Readiness for MTSS Implementation, guides them to analyze strengths and areas for improvement schoolwide as they prepare to implement MTSS. Chapter 16, Collaboration as the Foundation for Implementing MTSS, addresses collaboration between general and special educators, as well as collaboration between teachers and other stakeholders. Finally, Chapter 17, Implementing MTSS: Voices From the Field, offers perspectives from teachers and administrators about the real-life challenges—and rewards—of implementing MTSS/RTI to improve mathematics outcomes in middle school.

We hope that readers find this book a helpful resource in helping all students to succeed in middle school mathematics. Their success has never been more important!

References

Ball, D. L., Hill, H. C., & Bass, H. (2005, Fall). Knowing mathematics for teaching. *American Educator,* 14–46.

Banilower, E. R., Smith, P. S., Weiss, I. R., Malzahn, K. A., Campbell, K. M., & Weis, A. M. (2013). *Report of the 2012 National Survey of Science and Mathematics Education.* Chapel Hill, NC: Horizon Research.

Berch, D. B., Mazzocco, M. M. M., & Ginsburg, H. P. (Eds.). (2007). *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore, MD: Paul H. Brookes Publishing Co.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P.,. . . Japel, C. (2007). School readiness and later achievement. *Developmental Psychology, 43,* 1428–1446.

Gersten, R., Chard, D. J., Jayanthi, M., Baker, S. K., Morphy, P., & Flojo, J. (2009). Mathematics instruction for students with learning disabilities: A meta-analysis of instructional components. *Review of Educational Research, 79,* 1202–1242.

Hanushek, E. A., & Rivkin, S. G. (2006). *School quality and the black–white achievement gap* (NBER Working Paper No. 12651). Washington, DC: National Bureau of Economic Research.

Siegler R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., . . . Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697.

Steele, J. (2003). Children's gender stereotypes about math: The role of stereotype stratification. *Journal of Applied Social Psychology, 33,* 2587–2606.

Walker, E. N. (2007). Why aren't more minorities taking advanced math? *Education Leadership, 65*(3), 48–53.

SECTION I

Building Numeracy in Middle School Students

OVERVIEW: FOUNDATIONS FOR MEETING ALL STUDENTS' NEEDS

Our goal in writing this book is to provide meaningful resources to you—teachers, instructional coaches, and leaders—as a cohesive and comprehensive tool to support student success in middle school mathematics classes. Section I sets the stage for the remainder of the book. We start by defining algebra readiness in the middle grades in Chapter 1. Next, we describe how instruction and assessment can work together in a multi-tiered system of support (MTSS) to meet students' needs (Chapter 2). In Chapter 3, we illustrate approaches to making instruction and assessment accessible to all students. We hope this section is a useful resource to continue referring to as you make your way through the rest of the book. You will find that we refer to topics introduced in these three chapters throughout the remaining three sections.

The chapters in this section will help you answer the following questions:

1. *What does algebra readiness look like in my middle-school mathematics classroom?* When students work algebraically, they are generalizing their knowledge about numbers and operations to solve problems with unknown quantities. Research on how students learn mathematics highlights three key factors in becoming ready for algebra: 1) procedural fluency with whole numbers, 2) conceptual understanding of rational numbers, and 3) proficiency with rational number operations. In Chapter 1, we describe how students' knowledge and understanding of whole-number concepts and operations lay the foundation for their work with rational numbers. We illustrate how carefully designed instruction can support students' foundational knowledge and help them become ready for algebra.

2. *How can I help all students be ready for algebra?* All students in your mathematics classroom can be ready for algebra. Some students may need more intensive instructional support to reach this goal than others. MTSS is a framework that integrates instruction and assessment to help identify the intensity of instructional support

your students need to be ready for algebra. In Chapter 2, we introduce MTSS and preview the three tiers of instructional support that are typical within MTSS. We discuss how you can use assessment results to help guide your decision making. These concepts are discussed in considerably more detail in Sections II and III of the book.

3. *What is accessibility, and how can I make my instruction and assessments more accessible?* Differentiated instruction, accommodations, and modifications can be implemented to improve the accessibility of your instruction and assessment. In Chapter 3, we describe each of these approaches to improving accessibility, provide examples to help differentiate each approach, and discuss when you might consider using them. An important point to remember from this chapter is that decisions to use these approaches may have different implications for students' opportunities to learn the content. Moreover, accommodations and modifications are typically made by a team of people who are working to support an individual student (e.g., an individualized education program [IEP] team).

CHAPTER 1

Laying the Foundation for Algebra

What do you notice about these problems?

- A boy has 13 apples. Four apples are red. The rest are green. How many green apples does he have?
- There are red and green apples in each basket. The ratio of red apples to the total number of apples is 4:13. If a boy has one basket with 4 red apples, how many green apples does he have?

How are these problems different? How are these problems similar? Why is it that a typical middle school student would have no difficulty solving the first problem (and actually might think it is so easy that there must be a trick) but would struggle to solve the second problem?

The transition from working with concrete objects and scenarios in elementary school (often similar to the first problem) to working with abstract concepts like ratios in middle school (as in the second problem) poses a barrier for many students. For some students, this is when mathematics becomes "magical," not in the sense of fairy princesses making your wishes come true, but more in the sense of casting evil spells. Resilient students usually progress through the content in spite of the evil spell, often relying on their procedural proficiency (instead of their conceptual understanding) to succeed. Less resilient students get mired down in the trickery. This is the beginning of the end of their love of mathematics.

Why is this transition so challenging for some students? In this chapter, we describe the transition from concrete to abstract mathematics and the importance this transition plays in preparing students for algebra. We talk about the critical role of numeracy in helping your students successfully navigate this transition.

PAVING THE WAY FOR ALGEBRAIC REASONING: SETTING THE FOUNDATION IN EARLY MATHEMATICS

Without knowing it, many young students are proficient in working with algebraic concepts. We see examples like the one shown in Figure 1.1.

In the Figure 1.1 example, students are not only making the transition from their knowledge of concrete objects (i.e., the dog bones) to a symbolic representation of the object (the number 3), but also solving for an unknown. In the example shown in Figure 1.2, students are again associating concrete objects (i.e., rabbits and ears)

3 dog bones +  = 7 dog bones

How many bones go in the doghouse to make this true?

Figure 1.1. Sample word problem involving algebraic reasoning (solving for an unknown).

with symbolic representations, but they are also evaluating the relationship between two quantities. Because of the nature of these examples, we as teachers, parents, and tutors sometimes don't recognize the valuable connections these problems have to algebraic concepts.

Simply put, working algebraically means that your students can generalize their knowledge about numbers and operations to solve problems with unknown quantities. When students think algebraically, they can see relationships among quantities without the particular quantities being present in the problem. In the problem with

Complete this table.

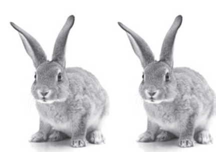
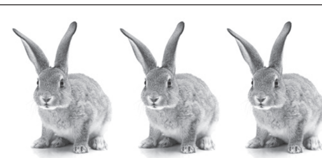
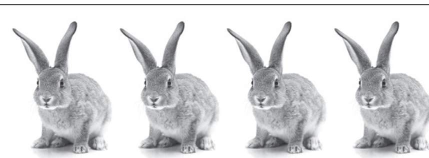
1 rabbit = 2 ears	
2 rabbits = ______ ears	
3 rabbits = ______ ears	
4 rabbits = ______ ears	

Figure 1.2. Sample word problem involving algebraic reasoning (evaluating the relationship between two quantities).

the dog bones and doghouse shown in Figure 1.1, the student is being asked to find an unknown quantity. Instead of this unknown being represented with a symbol (as is done with most variables in middle school and beyond), it is represented with a concrete object (the doghouse). However, students are still asked to generalize their arithmetic skills to find the number of bones in the doghouse. Students could find the value of the unknown by any number of means:

$$3 + 4 = 7$$
$$3 + 1 + 1 + 1 + 1 = 7$$
$$3 + 2 + 2 = 7$$
$$7 - 3 = 4$$
$$7 - 4 = 3$$
$$7 - 1 - 1 - 1 - 1 = 3$$

By providing opportunities for young students to think algebraically, we are teaching them to use variables and, as in the rabbit problem, they begin to see covariation among quantities (which serves as a pre-skill to understanding functions).

These examples are concrete in nature and allow students to use their understanding of numbers and operations in a flexible way. However, as soon as we represent the problem abstractly as $3 + x = 7$ and expect students to solve for x in a specific and precise manner, many students begin to stumble and have difficulties. To be more realistic, we understand that most middle school students would have little difficulty with this example. However, given a slightly more complex problem such as the grade 8 item from the 2011 National Assessment of Educational Progress (NAEP) shown in Figure 1.3, many students see no connection between this abstract representation and the concrete representations they worked with throughout elementary school.

The solution appears as impossibly magical as pulling a rabbit out of a hat.

For many students, the leap from concrete to abstract mathematical representations is often the cause of their difficulties with mathematics, and algebra in particular. However, strong numeracy skills can help students transition from working with concrete representations to abstract algebraic reasoning and help them navigate the mathematical magic.

WHAT ARE NUMERACY SKILLS?

In thinking about what skills and knowledge students need to successfully transition from concrete mathematics in elementary school to abstract mathematics in high school algebra, several notable organizations have contributed their perspectives.

12. The point $(4, k)$ is a solution to the equation $3x + 2y = 12$. What is the value of k?

    A. –3

    B. 0

    C. 2

    D. 3

    E. 4

Figure 1.3. Sample grade 8 item from the 2011 National Assessment of Educational Progress (NAEP).

The National Mathematics Advisory Panel (NMAP; 2008), a presidential panel convened to address issues of mathematics underachievement, identified several foundational skills that support students' algebra readiness, including fluency with whole numbers, fluency with fractions, and particular aspects of geometry and measurement. In this book, we highlight the importance of the first two foundational skills and identify three key areas to support students' readiness for algebra:

1. Procedural fluency with whole number operations
2. Conceptual understanding of rational number systems
3. Proficiency operating with rational numbers

These fundamental skills and knowledge can be seen as a progression that helps students move from concrete to abstract reasoning. First, students develop a conceptual understanding of whole numbers (i.e., 0, 1, 2, 3, . . .) and then they gain skills in adding, subtracting, multiplying, and dividing whole numbers. As they gain proficiency with whole number operations, they are better able to verify the reasonableness of their solutions. Next, they build on and extend their conceptual understanding of whole numbers to develop a conceptual understanding of rational numbers (i.e., any number that can be written as $\frac{p}{q}$, where p and q are integers). Then, students integrate their conceptual understanding of rational numbers with their proficiency in whole number operations to compute with rational numbers. Finally, students combine these skills and knowledge to generalize arithmetic principles learned with whole and rational numbers to solve abstract problems involving symbolic notation. This progression, illustrated in Figure 1.4, helps build a foundation for algebraic reasoning.

These skills combine to contribute to students' overall understanding of numbers, or numeracy. *Numeracy,* often called number sense, refers to a "child's fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons" (Gersten & Chard, 1999, pp. 19–20). You might think you have heard numeracy often referenced when talking about young students' development of mathematics skills. You are right. In fact, in the Common Core State Standards in Mathematics (CCSS-M), number sense is referenced as students develop the concept of whole numbers in grade 1 and then begin to understand fractions in grade 5. There is no mention of numeracy or number sense in the middle grades content standards.

However, if you look closer at most content standards, including the CCSS-M, you will see that middle school students are required to flexibly use numbers across number systems (whole numbers, integers, rational numbers). You will also notice that students need to use properties of operations lawfully and understand why they work. Students need to apply their knowledge to problem-solving scenarios to make predictions or solve the situation. You will see that students need to operate proficiently with whole numbers, integers, and rational numbers, and understand, justify, and evaluate outcomes of operations. These skills all relate to students' number sense. What's more, these also all relate to students' development of algebraic reasoning. In other words, even though numeracy is not explicitly mentioned within middle school content standards, it is implicit to students' being able to meet those standards.

This chapter focuses on the three interconnected concepts, listed previously, that support students' transition from concrete to abstract thinking and serve as the foundation for success in algebra: proficiency with whole number operations, focusing

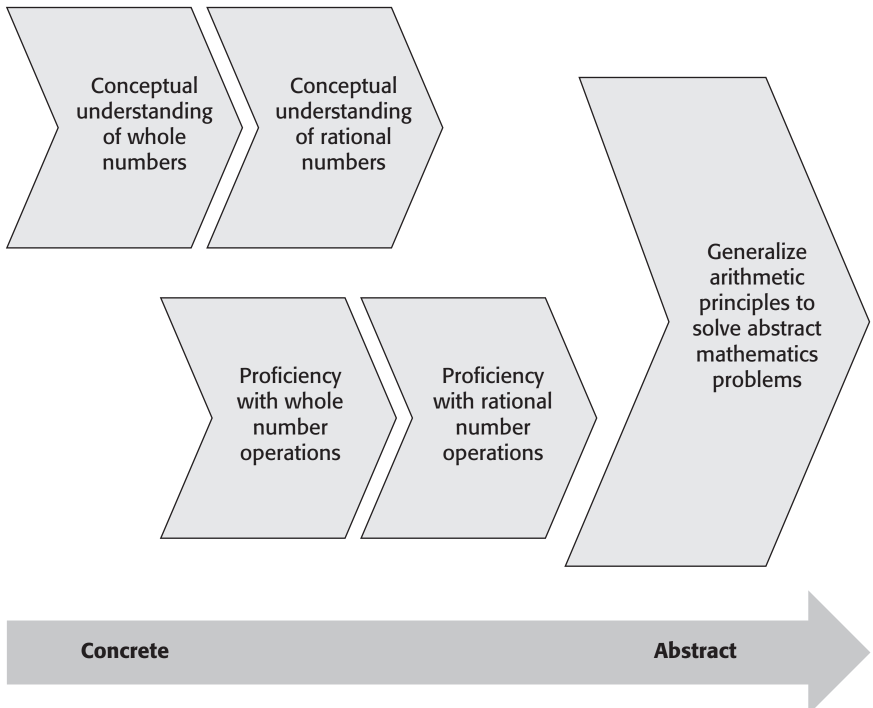


Figure 1.4. Progression of skills that help to build a foundation for algebraic reasoning.

specifically on students' ability to work with properties of operations; conceptual understanding of rational number systems; and proficiency with rational number operations.

Just as pillars support the structure of a building, we can regard these foundational skills as pillars serving to support students' algebraic reasoning. This relationship is illustrated in Figure 1.5. The subsections that follow describe what each pillar "looks like" when students demonstrate these skills in the classroom. Each subsection also highlights core mathematical concepts that middle school teachers can focus upon to strengthen each of the three pillars.

The First Pillar: Developing Proficiency With Whole Number Operations

To develop students' proficiency or procedural fluency with whole number operations, it is important for teachers to understand what is meant by *proficiency* or *procedural fluency*. The examples of Landon and Jailynn described in the following vignette illustrate what this might look like in the middle school classroom.

TWO SEVENTH-GRADERS: LANDON AND JAILYNN

Landon is a seventh-grader who is often referred to as a "math whiz." He has been practicing for the University Interscholastic League (UIL) Number Sense competition, held throughout the state of Texas, and is the best in his club. He can add, subtract, multiply,

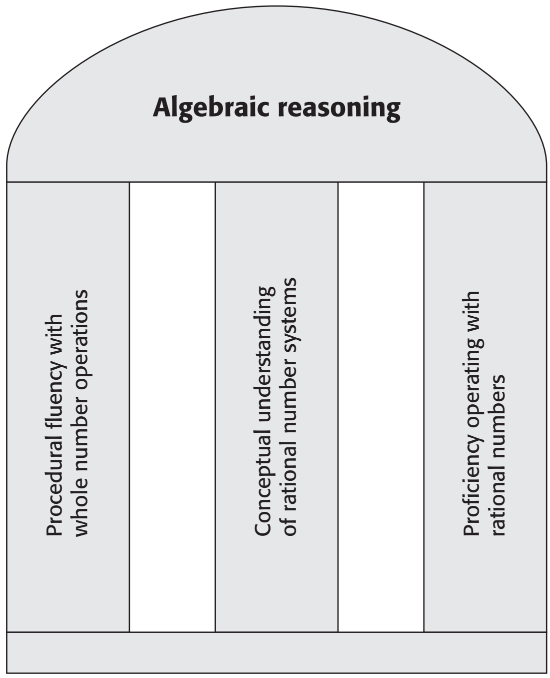


Figure 1.5. The three pillars of algebraic reasoning.

and divide two- and three-digit numbers in his head, but he struggles with word problems. Although he is doing well in his mathematics class, Landon often responds incorrectly to word problems on tests; he gets frustrated when trying to set up the problem and has difficulty deciding how to solve it.

Jailynn is a seventh-grader who does well in her mathematics class but doesn't think of herself as a "math person." She isn't able to execute complex algorithms in her head and is often the last to complete her mathematics tests. She likes word problems and is able to correctly translate the word problem to a symbolic problem and find the solution. She takes her time to complete her mathematics tests because she verifies her answers.

Which student would you say has greater procedural fluency? Although Landon can quickly execute operations and has developed mental arithmetic strategies, when given a mathematics problem in context, he struggles to select the appropriate operation and execute a strategy to solve it. Jailynn, on the other hand, may lack speed and the ability to perform complex mental arithmetic, but she understands the operations in novel contexts, effectively employs the algorithms, and can use different solution strategies to verify her answers. Both students possess unique but important aspects of procedural fluency.

As this vignette illustrates, defining procedural fluency as being able to quickly add, subtract, multiply, and divide is too narrow. In the 2001 publication *Adding It Up,* the National Research Council more expansively describes procedural fluency as the "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121).

Being procedurally fluent allows students to devote more of their attention to working out more complex problems, connecting the procedures with concepts, and seeing relationships among quantities. Also, arithmetic skills of upper elementary (Bailey, Siegler, & Geary, 2014; Hecht, Close, & Santisi, 2003) and middle school (Hecht, 1998) students significantly contribute to their ability to perform fraction computation.

Although many people would have said that Landon had greater procedural fluency than Jailynn because of the speed with which he computes as well as the mental strategies he employs, we can see from this definition that he lacks some of the other components of procedural fluency. Jailynn, who many people would have said was not procedurally fluent because she lacks speed and mental arithmetic strategies, has other skills that contribute to her proficiency with procedures. Although proficient in some aspects of whole-number operations, both of these students may encounter difficulties as they make the transition from concrete arithmetic to more abstract algebraic thinking. As a middle school mathematics teacher, you will teach students like Landon and Jailynn, and your task will be to help them both develop a deeper understanding of numbers, or numeracy, in order to strengthen their procedural fluency. Doing so involves working with properties of operations.

For middle school students, advancing their procedural fluency with whole numbers to the point where it will support their algebraic reasoning involves understanding and being able to apply basic properties of operations. The basic properties of operations that support algebra readiness include the distributive property, the commutative and associative properties of addition and multiplication, the identity elements for addition and multiplication, the inverse properties of addition and multiplication, and mathematical equality. Examples of these properties are shown in Table 1.1.

For Landon, understanding these properties of operations will help him understand the relationships among operations to be able to use them more flexibly when solving word problems. For Jailynn, building proficiency with these properties of operations will increase her procedural efficiency so she becomes faster and better able to compute using mental arithmetic strategies. In other words, working with properties of operations strengthens students' conceptual understanding and their procedural fluency.

In their 2008 publication on learning processes for NMAP, Geary and his colleagues described the importance of understanding properties of operations to help

Table 1.1. Properties of operations that support algebra readiness

Property	Example
Distributive	$4(2 + 3) = (4 \times 2) + (4 \times 3)$
Commutative	Addition: $5 + 7 = 7 + 5$ Multiplication: $6 \times 4 = 4 \times 6$
Associative	Addition: $(1 + 3) + 2 = 1 + (3 + 2)$ Multiplication: $(4 \times 5) \times 2 = 4 \times (5 \times 2)$
Identity	Addition: $7 + 0 = 0 + 7$ Multiplication: $7 \times 1 = 1 \times 7$
Inverse	Addition: $5 + (-5) = 0$ Multiplication: $6 \times \frac{1}{6} = 1$

students become procedurally proficient. Students who understand properties of operations can efficiently solve arithmetic problems, identify and correct errors, apply algorithms in contextualized settings, and generalize their understanding to novel situations. Also, as your students get better at using properties of operations to operate with whole numbers, they should be able to transfer their knowledge to solve problems with rational numbers as well as with symbols. This forms the foundation for their ability to lawfully manipulate numbers and symbols to solve algebraic problems.

Because properties of operations have been part of most elementary and middle school content standards for years, you might ask why we are emphasizing the importance of these skills now. Even though these skills are in the content standards, many textbooks and other instructional materials have done little to help students understand properties of operations beyond learning their definitions. In fact, state accountability tests often include items, like the one shown next, that test students' ability to label the property of operation correctly.

Which property is represented by this equation?

$(1 + 4) + 8 = 1 + (4 + 8)$

A. Associative property

B. Commutative property

C. Distributive property

D. Identity property

However, these items don't assess whether students can use the properties flexibly to solve problems. Two properties are particularly valuable for helping middle school students develop conceptual understanding and procedural fluency: mathematical equality and the distributive property.

Mathematical Equality: The Mortar Between the Bricks Perhaps the most important property that is often underemphasized in elementary and middle school mathematics is mathematical equality. Many teachers may take this property for granted and provide little instruction to students on its importance. However, students' knowledge and application of this property contributes to their understanding of the lawfulness of mathematics and can affect their ability to solve algebra problems. What is more, if students do not understand mathematical equality, they may continue to think of mathematics as mysterious magic that abides by made-up rules.

Understanding mathematical equality means that students see the equal sign as bridging equivalent relationships between expressions (Baroody & Ginsburg, 1983). Knowing that the equal sign indicates that the quantities are equivalent helps students understand the reasons for rules such as "If you do something to one side, you have to do the same thing to the other side." However, in the elementary school grades, the equal sign is often viewed as an operator symbol that directs students to do something. For example, if a teacher writes "46 – 14 =" on the board, most students would "do" the subtraction and produce the correct answer of 32. In these instances, students learn that the equal sign is a command that directs them to operate. Some students will either directly or indirectly assume that answers to problems such as these have one (and only one) correct answer, and that it must be a number. Solutions such as

Ways to represent 46 − 14

46 − 6 − 8

46 − 10 − 4

(46 + 4) − 14 − 4

(45 + 1) − (15 − 1)

Figure 1.6. Example of multiple ways to represent the solution to a problem.

46 – 10 – 4 or (45 + 1) – (15 – 1) would be discounted as incorrect. This type of instruction focuses often on what procedure students are to follow when they see the equal sign, rather than on developing students' conceptual understanding of what the equal sign means. This assumption will limit students' ability to think flexibly about quantities and manipulate numbers to solve algebra problems.

As a middle school mathematics teacher, you may find some students come to your classes with these assumptions ingrained in their thinking. They may argue and protest or think you are invoking more mathematical magic when you tell them that there are multiple ways to represent the solution to 46 – 14, as shown in Figure 1.6. To help students see the lawfulness of these solutions, you will likely need to design your instruction carefully to help students identify their misunderstandings and then work to reinforce the meaning of mathematical equality. (See the instruction chapters in Section II of this book for information on the importance of dispelling misconceptions [Chapter 4] and guidance on designing instruction to overcome misconceptions [Chapter 7].) Beginning first with whole number operations and then increasing the complexity by introducing variables, you can demonstrate that the meaning of mathematical equality remains constant when working with concrete to abstract representations. The reward for the hard work of learning this important property of operations will come when students can flexibly work with numbers to solve increasingly complex problems.

Distributive Property: Reinforcing the Pillar Another property of operations that is indispensable in algebra is the distributive property. Because students are regularly asked to employ the distributive property to solve for x in problems such as $2(3 + x) = 23$, it probably comes as no surprise that we are emphasizing its importance for success in algebra. Although solving these types of problems is important, there are many other reasons for emphasizing the distributive property in your instruction. Specifically, understanding this property can help students like Jailynn perform mathematical operations faster and more efficiently.

As students build their procedural fluency, some students may need additional help in developing strategies to increase their efficiency. For example, Jailynn struggles to carry out operations quickly and is not able to perform complex mental arithmetic. Her inefficiency may become a burden for solving increasingly complex problems in middle and high school mathematics classes. Learning to use the

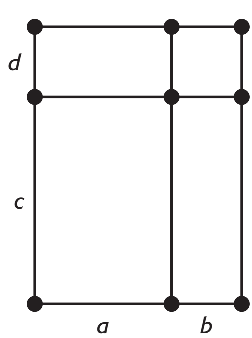
Table 1.2. Comparison between the FOIL method and the distributive property

FOIL method	Distributive property method
$(4 + x)(3 + x)$	$(4 + x)(3 + x)$
First terms: $(4)(3) = 12$	Distribute the 4 across the second expression: $(4)(3) = 12$
Outside terms: $(4)(x) = 4x$	$(4)(x) = 4x$
Inside terms: $(x)(3) = 3x$	Distribute the x across the second expression: $(x)(3) = 3x$
Last terms: $(x)(x) = x^2$	$(x)(x) = x^2$
Solution: $x^2 + 3x + 4x + 12 = x^2 + 7x + 12$	Solution: $x^2 + 3x + 4x + 12 = x^2 + 7x + 12$

distributive property may help her turn complex problems into simple arithmetic that she can easily solve in her head. Consider the problem 42 × 63. Using place value and the distributive property, this problem can be written as the sum of two expressions that are much less complex: [(42 × 6) × 10] + (42 × 3). If Jailynn is not ready to compute two-digit by one-digit multiplication in her head, the problem can be further decomposed into single digit multiplication: (40 × 60) + (2 × 60) + (40 × 3) + (2 × 3). Using the distributive property in this way can help students like Jailynn increase their speed in executing algorithms as well as develop strategies for mental computation.

Another important reason for having a thorough understanding of the distributive property is that it demystifies some of the "tricks" students learn to solve problems in algebra. For example, the FOIL method is routinely used to multiply binomials. The FOIL mnemonic represents the steps students take to multiply the **f**irst terms in each binomial, then the **o**utside terms, then the **i**nside terms, and then the **l**ast terms. Although this is technically correct, the FOIL method is nothing more than an application of the distributive property, as shown in Table 1.2.

As teachers clutter the curriculum with tricks like the FOIL method, students become less certain of which actions are lawful and begin to see mathematics as a series of seemingly random rules that are memorized and applied in special circumstances. Visual representations, like the one shown in Figure 1.7, can be used to help



$$(a + b) \times (c + d) = (a \times c) + (b \times c) + (a \times d) + (b \times d)$$

Figure 1.7. Example of a visual representation to help students understand how the distributive property works.

students conceptually understand why the distributive property works as it does. Conceptually understanding the distributive property addresses both potential gaps in applying the procedures and improves efficiency. This may be particularly helpful for students who see the distributive property as a set of rules that must be followed in a certain order.

Although we have highlighted mathematical equality and the distributive property here, students' understanding of other properties of operations is an important component of numeracy that will help them develop proficiency with whole number operations and, ultimately, to apply algorithms to solve algebraic problems. As students become proficient in using properties of operations with numeric representations, they can generalize their knowledge to solve increasingly more abstract problems in algebra.

The Second Pillar: Understanding Rational Numbers Conceptually

Young children understand the meaning of numbers from a very young age, and—even without knowing it—they have a firm grasp of concepts such as cardinality ("I got two cookies from Ms. Robinson") and even the ordinal meaning of numbers ("I came in first place in the race"). Quickly, they begin to understand concepts such as quantity ("I have a lot of cookies") and can begin to make quantity comparisons ("No fair! You got more cookies than me"). Soon, an understanding of number as a distance between points develops ("I can jump over three boxes") and distance comparisons ("It is taking forever to get to Grandma's. Are we there yet?"). In each of these instances, children's conceptual understanding of whole numbers is rooted in concrete experiences, objects, or representations.

Once schooling starts, students begin to formalize their understanding of natural numbers and then extend this understanding to whole numbers. They understand that numbers represent quantities with magnitude. They understand that equivalent representations of numbers have the same quantity. Although understanding natural and whole numbers lays the foundation for students to perform operations and then later develop a conceptual understanding of integers and rational numbers, an essential ingredient to this mix is students' understanding of the concept of place value. *Place value* is the value of a digit in a base-10 system and is typically referenced as a shorthand notation for writing numbers. Connections among these different conceptual understandings in mathematics are illustrated in Figure 1.8.

Not only does having a foundational knowledge of place value help students understand basic algorithms and develop greater procedural efficiency, it also serves as a conceptual and procedural link between number systems. Conceptually, students can use their understanding of place value to see how rational numbers are quantities with magnitude. Procedurally, students can use their understanding of place value to help them generalize their knowledge of operations with whole numbers to operations with rational numbers.

Consider the two problems shown in Figure 1.9. Although both items assess the same grade 5 content standard from the CCSS-M that states "read, write, and compare decimals to thousandths" (5.NBT.3), they are tapping into different dimensions of students' understanding. The item on the left assesses students' knowledge of place value vocabulary (hundredths place) and ability to recognize a specific value within a given number. The item on the right assesses students'

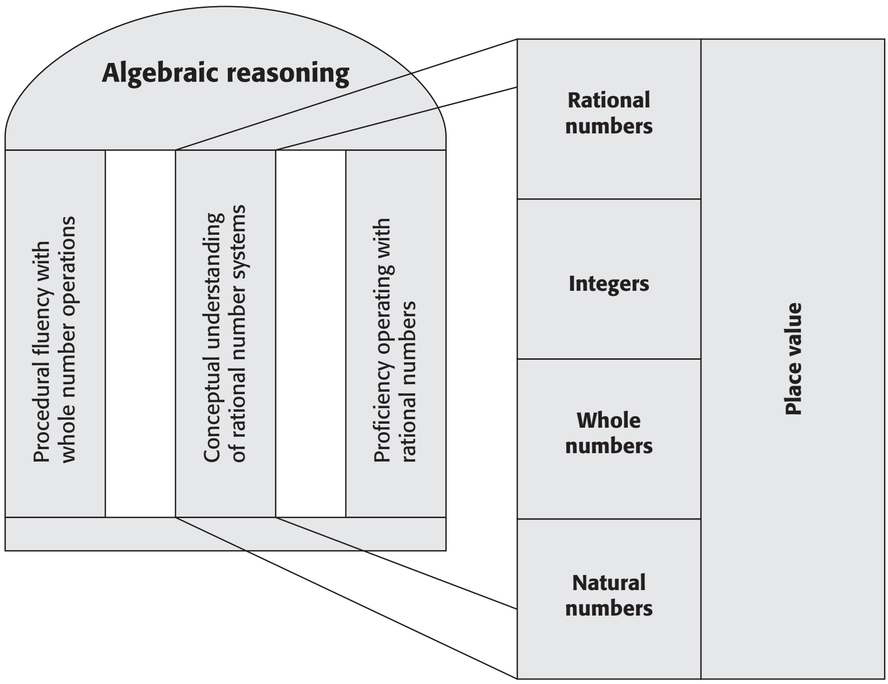


Figure 1.8. Connections among different conceptual understandings in mathematics: natural and whole numbers, integers and rational numbers, and place value.

conceptual understanding of place value by asking students to identify the value of each digit within a given number as well as identify how the digits relate to each other to form the number. Also, the students' knowledge of place value with whole numbers is integrated into their knowledge of place value with decimals. Constructing learning and assessment opportunities that integrate these important dimensions of place value provides the foundation for understanding rational numbers.

As a middle school teacher, you know the struggles many students have when it comes to learning about rational numbers, particularly when learning about fractions. Even for students who have been historically successful in mathematics, learning fractions can be perplexing, vexing, and downright maddening. Students commonly make many generalizations when learning about natural and whole

Which digit is in the hundredths place in 536.184? Answer: 8	What is the value of 8 in 536.184? Answer: $8 \times \left(\frac{1}{100}\right)$

Figure 1.9. Comparison of two fifth-grade test items assessing different dimensions of students' understanding of decimals and related concepts.

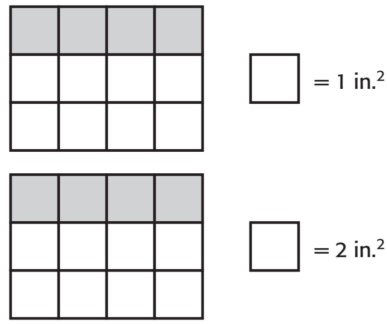


Figure 1.10. Example of a shaded figure used to show that the principle "size always matters" is a misconception.

numbers that can serve as roadblocks for learning fractions—for example, consider the following:

- *Misconception 1:* "Size always matters." Although this is the case for whole numbers, it does not always hold true for rational numbers. With whole numbers, size matters. For example, the area of the two shapes in Figure 1.10 is different because the size of the unit square is different. However, when talking about fractions, the proportion of the two shapes that is shaded is the same.
- *Misconception 2:* "Bigger numbers are bigger." With whole numbers, "bigger" numbers have greater quantity (13 is "bigger" than 2). However, when talking about fractions, "bigger" denominators indicate smaller quantities ($\frac{1}{13}$ is "smaller" than $\frac{1}{2}$).
- *Misconception 3:* "Multiplying makes numbers bigger." When multiplying whole numbers, the product is a larger number than the factors ($2 \times 2 = 4$). However, when multiplying proper fractions, the product is a smaller number ($\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$).
- *Misconception 4:* "Dividing makes numbers smaller." When dividing whole numbers, the quotient is a smaller number than the dividend ($15 \div 5 = 3$). However, when dividing proper fractions, the quotient is a larger number than the dividend ($\frac{1}{15} \div \frac{1}{5} = \frac{1}{3}$).

Each of these overgeneralizations implies that students do not conceptually understand rational numbers. In some cases, students' previous instructional experience with whole numbers or fractions has caused some of the confusion. For example, instruction that overly relies on fraction models such as pizzas or pies can limit students' understanding of the meaning of fractions. Relying too heavily on circular models may cause students to believe that fractions are always shaded parts of circles. Students may not make the connection that the model represents a numerical value unless other representations are presented to them (e.g., number lines). When students conceptually understand rational numbers, they understand that rational numbers represent quantity with magnitude. They understand that rational numbers have multiple representations (including fractions and decimals) but require equal partitioning of a whole or set.

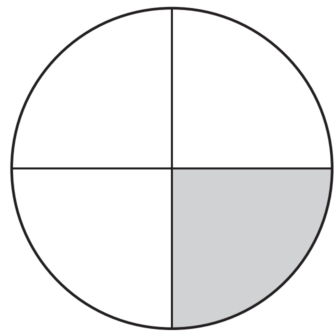


Figure 1.11. Representation of $\frac{1}{4}$ representing understanding of equal partitioning of a whole.

Consider how students develop and demonstrate conceptual understanding of fractions, using the example of $\frac{1}{4}$ to illustrate the progression. Initially, many students would generate a representation like the one shown in Figure 1.11.

Although this representation indicates an understanding of equal partitioning of a whole, it does not represent a full understanding of the meaning of fractions. As students gain greater awareness that fractions can represent equal partitioning of a set, they might represent $\frac{1}{4}$ as shown in Figure 1.12.

Still, to understand fractions conceptually, students should know that fractions are numbers with magnitude that can be used to measure quantities. Representing the fraction $\frac{1}{4}$ using a number line would indicate a deeper conceptual understanding, such as the representation in Figure 1.13.

By representing a fraction as a point on a number line, students recognize a fraction as a quantity, or distance from zero, and see the meaning of equal partitioning of a number line. Students begin to compare the quantity of fractions and further develop and refine the mental number line they constructed for whole numbers in elementary school. They also begin to understand that whole numbers can be represented as fractions and that fractions can be greater than or less than 1, as well as less than 0.

As students understand the magnitude of a fraction, students would represent the fraction $\frac{1}{4}$ using the type of representation shown in Figure 1.14.

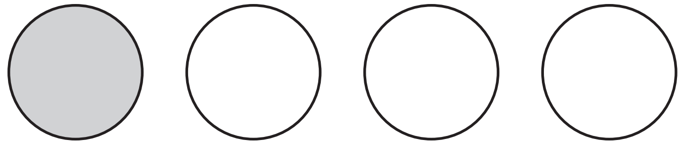


Figure 1.12. Representation of $\frac{1}{4}$ representing understanding of equal partitioning of a set.

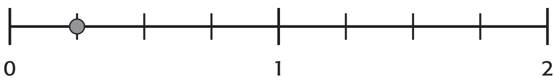


Figure 1.13. Representation of $\frac{1}{4}$ representing understanding that fractions are numbers with magnitude that can be used to measure quantities.

As students generate increasingly more sophisticated representations of fractions (as depicted in this progression), they demonstrate deep conceptual understanding that includes recognizing fractions as quantities with magnitude, and they understand the importance of equal partitioning. Students can then integrate their conceptual understanding of rational numbers with their knowledge of natural and whole number systems to increase their flexibility when working with these numbers. For example, students can use composition and decomposition to reason about equivalent fractions, and similarly, decimals. Imagine that a student, Matt, conceptually understands the quantity $\frac{1}{2}$. He should be able to recognize that $\frac{1}{4}+\frac{1}{4}=\frac{2}{4}$ is an equivalent representation to $\frac{1}{2}$ because the magnitude of the representation has not changed even if the quantity is divided into more equal parts. An equivalent fractions chart like the one in Figure 1.15 is often used to teach students about equivalent fractions. Although this chart can be used as a quick reference, students shouldn't need to memorize this chart if they have a solid conceptual understanding of fractions as quantities with magnitude.

In summary, students' understanding of natural and whole number systems—place value, in particular—supports their conceptual understanding of rational numbers. In turn, this conceptual understanding of rational numbers lays the foundation for success in future mathematics. In particular, conceptual understanding of rational numbers has been found to significantly contribute to upper elementary (Hecht et al., 2003) and middle school (Hecht, 1998) students' ability to operate and estimate with fractions as well as students' ability to set up word problems.

The Third Pillar: Developing Proficiency With Rational Number Operations

As we mentioned, many middle school students struggle with fractions. Their limited conceptual understanding is often observed in their confusion with fraction operations. Researchers have found that a strong understanding of foundational fraction concepts predicts fluency with fraction operations. However, we often do not know that students struggle with basic fraction concepts until we get to operations.

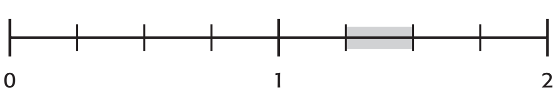


Figure 1.14. Representation of $\frac{1}{4}$ representing understanding of the magnitude of a fraction.

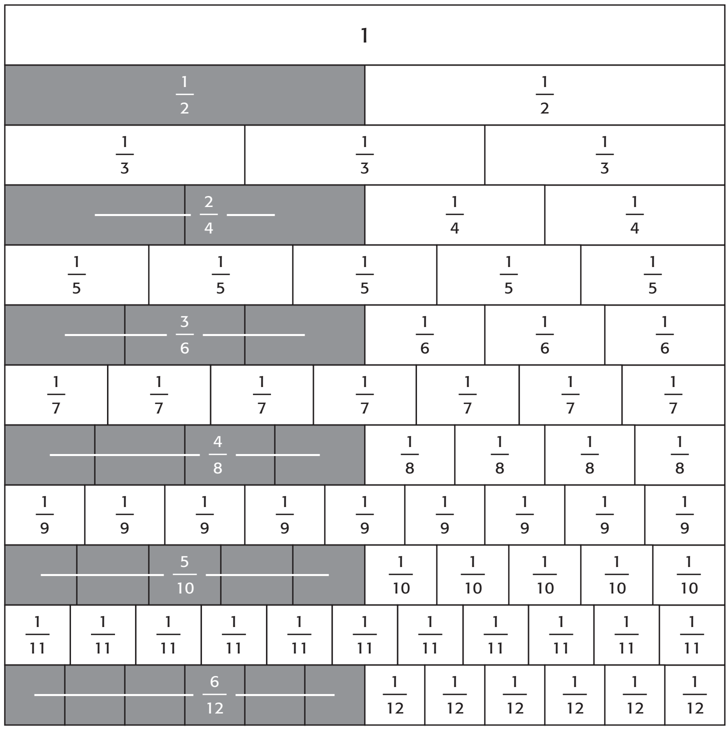


Figure 1.15. Equivalent fractions chart.

Students develop proficiency with fractions in a number of different ways. Hallett, Nunes, and Bryant (2010) found at least five. Some students have stronger conceptual knowledge and weaker procedural knowledge; others have stronger procedural knowledge and weaker conceptual knowledge. Some have average amounts of both, whereas others have strong knowledge of both. This information seems intuitive, but researchers found something interesting when they looked at students' ability to solve fraction problems and reason quantitatively. Students with stronger conceptual understanding of fractions outperformed students with average procedural knowledge, but students with stronger procedural knowledge did not outperform students with average conceptual knowledge. In other words, higher levels of conceptual understanding may help students compensate for average procedural knowledge of fractions, but higher levels of procedural knowledge may not have the same effect on performance.

Conceptual understanding of fractions also extends to conceptually understanding the algorithms that govern operations with fractions. Because operations with fractions may seem counterintuitive to many students, grounding instruction in fraction concepts and the underlying mathematical rationale for the algorithm may help students see through the magic's smoke and mirrors to understand the meaning of the operations. Once students understand the lawfulness of the algorithms, they can begin to see how the procedures can be applied in general. This is an important link to algebra.

Dividing fractions is one of the most vexing of the operations. First, many students (and adults) have a difficult time explaining a situation that involves division of fractions. Textbooks often provide a limited number of situations that can be conveniently modeled using ribbon or string. Although these models help introduce students to the algorithm and provide some context for developing conceptual understanding, they often focus on the measurement model of division (and do little to develop the partitive model or the product-and-factors model) and leave students with an incomplete picture of division of fractions.

> Strong conceptual understanding of fractions may compensate for weaker procedural fluency, but the converse may not be true.

To demonstrate how to build conceptual understanding of the meaning of division of fractions, we provide, in Figure 1.16, an example of how teachers can build on students' conceptual understanding of fractions to develop the meaning of division of fractions. This example is not intended to serve as an instructional sequence,

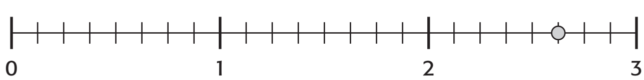
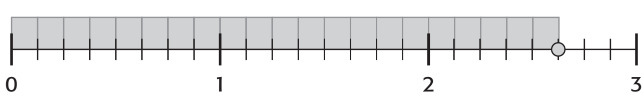
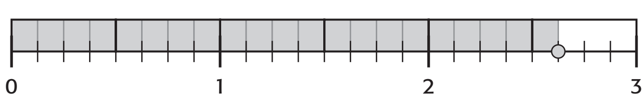
Problem: Vanessa has $2\frac{5}{8}$ feet of rope. She wants to cut the rope into $\frac{1}{2}$ foot sections. How many $\frac{1}{2}$ foot sections of rope will she have?	
What is the problem asking?	How many $\frac{1}{2}$ units of length are in $2\frac{5}{8}$?
Model $2\frac{5}{8}$ on a number line	
Add a fraction model of $2\frac{5}{8}$ to the number line	
Divide the model into $\frac{1}{2}$ units	
Determine the number of $\frac{1}{2}$ units in $2\frac{5}{8}$	There are five $\frac{1}{2}$ units and $\frac{1}{4}$ of a $\frac{1}{2}$ unit. Therefore, $2\frac{5}{8} \div \frac{1}{2} = 5\frac{1}{4}$.

Figure 1.16. Example of how teachers can build on students' conceptual understanding of fractions to develop the meaning of division of fractions.

Table 1.3. Step-by-step discussion of the algorithm for division of fractions

Properties of operations	Algebraic examples	Numeric examples
Division is the inverse operation of multiplication	$M \div N = X\ (N \neq 0) \Leftrightarrow M = X \times N\ (N \neq 0)\ (1)$	$32 \div 10 = x$ $32 = x \times 10$
Given	Put $M = \frac{a}{b}$, $N = \frac{c}{d}$, and $X = \frac{x}{y}$ $(b \neq 0, d \neq 0, y \neq 0)$	$M = \frac{7}{16}$ $N = \frac{3}{8}$ $X = \frac{x}{y}$
Substitution	$\frac{a}{b} \div \frac{c}{d} = \frac{x}{y}$	$\frac{7}{16} \div \frac{3}{8} = \frac{x}{y}$
Fundamental theorem of fractions	$\xrightarrow{by(1)} \frac{a}{b} = \frac{x}{y} \times \frac{c}{d}$	$\frac{7}{16} = \frac{x}{y} \times \frac{3}{8}$
Identity property of multiplication	$\xrightarrow{\text{multiply both sides by } \frac{d}{c}} \frac{a}{b} \times \frac{d}{c} = \frac{x}{y} \times \frac{c}{d} \times \frac{d}{c}$	$\frac{7}{16} \times \frac{8}{3} = \frac{x}{y} \times \frac{3}{8} \times \frac{8}{3}$
Multiplicative inverse	$\xrightarrow{\frac{c}{d} \times \frac{d}{c} = 1} \frac{a}{b} \times \frac{d}{c} = \frac{x}{y}$	$\xrightarrow{\frac{3}{8} \times \frac{8}{3} = 1} \frac{7}{16} \times \frac{8}{3} = \frac{x}{y}$
If $a = b$ and $b = c$, then $a = c$	$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$	$\therefore \frac{7}{16} \div \frac{3}{8} = \frac{7}{16} \times \frac{8}{3}$

Source: Ketterlin-Geller & Chard (2011).

but instead to illustrate the integration of students' conceptual understanding of fractions.

For many students, the second troubling aspect of dividing fractions is the algorithm. Although the algorithm (affectionately called "invert and multiply") is straightforward and easy to execute, many students do not understand why or how it works—further adding to the magic and mystery. Having a conceptual understanding of the meaning of division of fractions is important, but students also need to understand the mathematical rationale for the algorithm and why it is lawful.

To demonstrate the lawfulness of the algorithm, we provide a step-by-step dissection of the algorithm in Table 1.3, along with a numerical example and a generalized example that includes symbolic notation. We have associated the appropriate properties of operations to each step to indicate the lawfulness of the processes. Also, by showing students the properties of operations, teaching this algorithm reinforces to them that the rules of arithmetic generalize from whole numbers to fractions, and that no tricks are being introduced. Again, this is not intended to serve as an instructional guide but should be discussed or reviewed with students to verify their conceptual understanding of the algorithm.

Algorithms for operations with fractions need to be grounded in students' conceptual understanding of fractions but also need to be taught conceptually. Conceptual approaches to teaching the meaning of operations with fractions and the mechanics of the operations may support subsequent fraction problem-solving skills and advanced quantitative reasoning skills necessary for algebra.

BRINGING IT ALL TOGETHER

Although we presented three pillars of numeracy as separate supports to build algebraic proficiency, no pillar can do the job alone. What's more, one pillar cannot be built in isolation from the others. Instead, these three pillars of numeracy need to be taught as integrated concepts to build a deeper level of mathematical proficiency.

Students' development of conceptual understanding begins as they work with number systems, properties, and operations. As students understand and use their numeracy, basic and advanced, it demystifies algebra and allows them to see that algebra is a way of using the knowledge they have learned across the number systems. Foundational understanding of how number systems relate, what lawful properties can be depended on across the systems, and why and how the operations can be used should be developed and strengthened as new number concepts and properties are introduced. This level of numeracy helps students develop algebraic reasoning and supports their lawful application of skills and knowledge to solving abstract problems. Given that algebraic reasoning is essential to college and career readiness, it is critical that students have a solid foundation for algebra.

SUMMARY: THE PILLARS OF ALGEBRAIC REASONING

This chapter identified three pillars of algebraic reasoning: 1) procedural fluency with whole number operations, 2) conceptual understanding of rational number systems, and 3) proficiency operating with rational numbers. These pillars serve as the foundation of algebraic reasoning. We also discussed the importance of mathematical equality and properties of operations. These ideas support the pillars and enable students to be flexible and efficient with numbers. Conceptual understanding is central to all of these ideas.

ADDITIONAL RESOURCES

You may wish to consult the following resources to learn more about the topics discussed in this chapter.

Geary, D. C., Boykin, A. W., Embretson, S., Reyna, V., Siegler, R., Berch, D. B., & Graban, J. (2008). *Chapter 4: Report of the Task Group on Learning Processes.* Retrieved from http://www.ed.gov/about/bdscomm/list/mathpanel/report/learning-processes.pdf

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Retrieved from https://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., . . . Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide* (NCEE #2010-4039). Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/fractions_pg_093010.pdf